Analysis of Wave Propagation in Inhomogeneous Optical Fibers Using a Variational Method

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Abstract—**The variational method is used to determine the propagation characteristics of an optical fiber consisting of a core with an arbitrary refractive-index distribution and a uniform cladding.**

The problem is first translated into a variational problem; the functional is computed upon the TEM-approximation basis. The variational problem is then solved by using the Rayleigh–Ritz method. The computed propagation characteristics are presented for refractive-index distributions of practical interest. The single-mode condition for a quadratic self-focusing fiber is obtained as $v < 3.53$ where v denotes the conventionally used normalized frequency; this result agrees with the numerical analysis by Dil and Blok. The obtained characteristic equations for the simplest case (uniform core case) are compared with analytic solutions to ensure the validity of the analysis.

I. INTRODUCTION

THE glass fiber is the most promising optical waveguide for use in the future optical communication and optical information-processing systems. It consists of a core having uniform or graded refractive index and a cladding with a smaller refractive index. When a glass fiber is produced by the double-crucible process [1], the boundary between the core and cladding becomes more or less gradual due to the diffusion of the constituting materials. Therefore, it is of practical interest to develop the method of analyzing the propagation characteristics of "continuously inhomogeneous" dielectric waveguides.

Kurtz and Streifer [2] analyzed the propagation of electromagnetic waves in a radially inhomogeneous waveguide. Snyder [3] used the perturbation theory to obtain the expression for the propagation constants of optical waveguides with diffused boundaries; his method is very simple and useful in many practical cases. Clarricoats and Chan [4] used a staircase function to approximate the continuous variation of the refractive index, and solved the problem in each stratified cylindrical medium. Kirchhoff [5], and Dil and Blok [6] used the power-series expansion to express the transverse field components in the inhomogeneous core. Heyke and Kuhn [7] computed the dispersion characteristics upon the basis of the variational expression of the propagation constant. However, they assumed a field distribution corresponding to a "reference permittivity profile," which is not always equal to the actual profile.

In this paper we present a more exact variational-method analysis of the radially inhomogeneous fiber. The scalar wave equation for the transverse field components and the boundary conditions are translated into a corresponding variational problem having the original equation as its Euler equation. Solving this variational problem by the Rayleigh–Ritz method, we obtain the characteristic equation giving the propagation constant. Numerical results are also presented for various refractive-index profiles of practical interest.

In Section II, basic equations are derived and formulated into a variational form. In Section III, the variational problem is solved by using the Rayleigh–Ritz method to give the characteristic equation; its solutions are shown in Section IV for several refractive-index profiles. The results of the analysis are compared with previously reported ones in Section V.

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II. BASIC EQUATIONS

A. Field Equation and Boundary Condition

Consider a radially inhomogeneous dielectric cylinder of radius a surrounded by a homogeneous dielectric medium (Fig. 1). We assume that the permittivity varies in the radial direction r as

$$\epsilon(r) = \epsilon_1[1 - f(r)], \qquad (0 \leq r \leq a) \tag{1}$$

where ϵ_1 is the permittivity upon the axis and $f(r)$ is a function satisfying

$$f(0) = 0 \qquad \text{and} \qquad |f(r)| \ll 1. \tag{2}$$

As has been shown by Kurtz and Streifer [2], the transverse electric and magnetic fields consist of two independent components:

$$\boldsymbol{E}_t = C_1\boldsymbol{E}_t^{(1)} + C_2\boldsymbol{E}_t^{(2)} \tag{3a}$$

$$\boldsymbol{H}_t = C_1\boldsymbol{H}_t^{(1)} - C_2\boldsymbol{H}_t^{(2)} \tag{3b}$$

where C_1 and C_2 are arbitrary constants, $\boldsymbol{E}_t^{(i)}$ and $\boldsymbol{H}_t^{(i)}$ $(i = 1,2)$ are given in terms of the "transverse field functions" $\Psi^{(i)}$ as

$$\boldsymbol{E}_t^{(i)} = \frac{(\epsilon_1/\epsilon_0)^{-3/4}}{k_0} (\pm j\boldsymbol{u}_r - \boldsymbol{u}_\theta)\Psi^{(i)}(r) \tag{4a}$$

and

$$\boldsymbol{H}_t^{(i)} = \frac{(\epsilon_1/\epsilon_0)^{1/2}}{\eta_0} \boldsymbol{u}_z \times \boldsymbol{E}_t^{(i)}, \qquad (i = 1,2). \tag{4b}$$

In these equations $j = (-1)^{1/2}$, $k_0 = \omega(\epsilon_0\mu_0)^{1/2}$, $\eta_0 = (\mu_0/\epsilon_0)^{1/2}$, ω is the angular frequency, and $\boldsymbol{u}_r$, $\boldsymbol{u}_\theta$, and $\boldsymbol{u}_z$ are the radial,

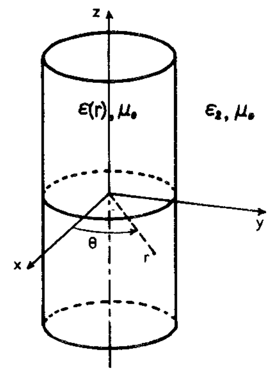

Fig. 1. Geometry of the waveguide. Symbols μ_0 and ϵ_2 denote the permeability of the media and the permittivity of the cladding.

azimuthal, and axial unit vectors, respectively. In (4) and also in the following equations, the upper and lower signs correspond to $i = 1$ and $i = 2$, respectively. The transverse field function satisfies the second-order differential equation [2]

$$\frac{d}{dr}\left(r\frac{d\Psi^{(i)}}{dr}\right) + \left[\omega^2\epsilon(r)\mu_0 - \beta^2 - \frac{(n \pm 1)^2}{r^2}\right] r\Psi^{(i)} = 0, \qquad (i = 1,2) \quad (5)$$

where β is the propagation constant and n is the azimuthal mode number $(\Psi^{(i)} \propto \exp(-jn\theta))$.[1]

The axial electric and magnetic fields are given in terms of $\Psi^{(i)}$:

$$E_z = (\epsilon_1/\epsilon_0)^{-1/4}(-C_1G^{(1)} + C_2G^{(2)}) \quad (6a)$$

and

$$H_z = j((\epsilon_1/\epsilon_0)^{1/4}/\eta_0)(C_1G^{(1)} + C_2G^{(2)}) \quad (6b)$$

where

$$G^{(i)} = -\frac{1}{\omega^2\epsilon_1\mu_0}\left[\frac{d\Psi^{(i)}}{dr} + \frac{(1 \pm n)}{r}\Psi^{(i)}\right], \qquad (i = 1,2). \quad (7)$$

The continuity conditions for $\Psi^{(i)}$ and $G^{(i)}$ at the core–cladding interface $(r = a)$, are expressed approximately[2] as

$$\left[\frac{1}{\Psi^{(i)}}\frac{d\Psi^{(i)}}{dr}\right]_{r=a} = \left[\frac{1}{\Psi_{\text{clad}}^{(i)}}\frac{d\Psi_{\text{clad}}^{(i)}}{dr}\right]_{r=a} \quad (8)$$

where $\Psi_{\text{clad}}^{(i)}$ $(i = 1,2)$ are the transverse field functions in the cladding, and are the solutions of

$$\frac{d}{dr}\left(r\frac{d\Psi_{\text{clad}}^{(i)}}{dr}\right) + \left[\omega^2\epsilon_2\mu_0 - \beta^2 - \frac{(n \pm 1)^2}{r^2}\right] r\Psi_{\text{clad}}^{(i)} = 0, \qquad (i = 1,2). \quad (9)$$

If we write

$$m = \begin{cases} n + 1, & i = 1 \\ & \text{for} \\ n - 1, & i = 2 \end{cases} \quad (10)$$

we may use common notations: $R(r)$ for $\Psi^{(1)}(r)$ and $\Psi^{(2)}(r)$, and $R_{\text{clad}}(r)$ for $\Psi_{\text{clad}}^{(1)}(r)$ and $\Psi_{\text{clad}}^{(2)}(r)$, to obtain

$$\frac{d}{dr}\left\{r\frac{dR(r)}{dr}\right\} + \left[\omega^2\epsilon(r)\mu_0 - \beta^2 - \frac{m^2}{r^2}\right] rR(r) = 0 \quad (11)$$

$$\frac{dr}{dr}\left\{r\frac{dR_{\text{clad}}(r)}{dr}\right\} + \left[\omega^2\epsilon_2\mu_0 - \beta^2 - \frac{m^2}{r^2}\right] r_{\text{clad}}(r) = 0 \quad (12)$$

where the boundary condition is given as

$$\left[\frac{1}{R(r)}\frac{dR(r)}{dr}\right]_{r=a} = \left[\frac{1}{R_{\text{clad}}(r)}\frac{dR_{\text{clad}}(r)}{dr}\right]_{r=a}. \quad (13)$$

Equations (11)–(13) are the starting equations of the following analysis.

B. Formulation of the Problem in a Variational Form

We first compute the fields in the core. Since the right-hand side of (13) is unknown, we tentatively express it by ϕ_β, which will be determined later in Section III-C.

The solution of (11) may be obtained also as the solution of the variational problem [8] to minimize the functional:

$$I[R] = R^2(a)a\phi_\beta - \int_0^a \left\{\frac{dR(r)}{dr}\right\}^2 r\,dr + \int_0^a \left[\omega^2\epsilon(r)\mu_0 - \beta^2 - \frac{m^2}{r^2}\right] R^2(r)r\,dr. \quad (14)$$

The justification of the above statement is given as follows. We assume that $I(R)$ is minimized for $R(r) = R_0(r)$, and consider a slightly deviated case in which $R(r) = R_0(r) + \delta\eta(r) \triangleq R_\delta(r)$, where $\eta(r)$ is an arbitrary continuous function of r and δ denotes a real small quantity. Putting $R_\delta(r)$ into (14) and considering that $I[R_\delta]$ must be minimized for $\delta = 0$, we obtain

$$\left.\frac{\partial}{\partial\delta}I[R_\delta]\right|_{\delta=0} = 2R_0(a)a\phi_\beta\eta(a) - 2\int_0^a r\frac{dR_0}{dr}\frac{d\eta}{dr}\,dr + 2\int_0^a \left[\omega^2\epsilon(r)\mu_0 - \beta^2 - \frac{m^2}{r^2}\right] rR_0(r)\eta(r)\,dr = 0. \quad (15)$$

By partially integrating the second term, we may write

[1] The two components, $i = 1$ and $i = 2$, correspond to two circularly polarized waves in opposite directions as seen in (4a). Hence, essentially, they are on an equal footing. The differences in the equations for the two components stems from the fact that in deriving (5), the azimuthal dependence is not assumed as stationary (sinusoidal) but as $\exp(-jn\theta)$.

[2] Suppose we define a parameter $\delta = 1 - \epsilon_2/\epsilon(a)$ expressing the relative discontinuity of the refractive index at the core–caldding interface. Then (8) is valid exactly only when $\delta = 0$, that is, when no step is present at the interface. When $\delta \neq 0$ as in case A shown in Fig. 2, the exact boundary conditions [10] are much more complicated than (8); some error will be included in the results obtained upon the basis of (8). However, in most practical cases we may approximate $\delta \to 0$ (weak guidance approixmation) and use (8).

$$\eta(a)aR_0(a)\left\{\phi_\beta - \left[\frac{1}{R_0}\frac{dR_0}{dr}\right]_{r=a}\right\} + \int_0^a \left\{\frac{d}{dr}\left(r\frac{dR_0}{dr}\right) + \left[\omega^2\epsilon(r)\mu_0 - \beta^2 - \frac{m^2}{r^2}\right]rR_0\right\}\eta(r)\,dr = 0. \quad (16)$$

Since $\eta(r)$ is an arbitrary function of r, this equation shows that $R_0(r)$ satisfies both the wave equation (11) and the boundary condition (13).

The final functional to be dealt with is obtained by putting (1) into (14):

$$I[R(r)] = a\phi_\beta R^2(a) + [\omega^2\epsilon_1\mu_0 - \beta^2]\int_0^a R^2(r)r\,dr - \int_0^a \left[\left\{\frac{dR(r)}{dr}\right\}^2 + \frac{m^2}{r^2}R^2(r)\right]r\,dr - \omega^2\epsilon_1\mu_0\int_0^a f(r)R^2(r)r\,dr. \quad (17)$$

III. SOLUTION OF THE VARIATIONAL PROBLEM BY RAYLEIGH–RITZ METHOD

A. Expansion by Ortho-Normalized Functions

To solve the above equation by using the Rayleigh–Ritz method [8], we express $R(r)$ in terms of a set of orthogonal functions. We consider two cases: $m \neq 0$ and $m = 0$, separately.

1) Case $m \neq 0$ ($n \geq 0$ for $i = 1$, or $n \geq 2$ for $i = 2$): In this case we use the orthogonal functions

$$g_{m,k}(r) = \frac{\sqrt{2}}{a}\frac{J_m(\lambda_k r)}{J_m(\lambda_k a)} \quad (18)$$

as the bases of the trial function of the Rayleigh–Ritz method. In (18) $\lambda_k = j_{m-1,k}/a$, where $j_{m-1,k}$ denotes the nonzero kth root of $J_{m-1}(z) = 0$. These functions satisfy the ortho-normalizing condition [9]:

$$\int_0^a g_{m,k}(r)g_{m,l}(r)r\,dr = \delta_{kl} \quad (19)$$

where δ_{kl} is Kronecker's delta.

2) Case $m = 0$ ($n = 1$, $i = 2$): In this case we denote the kth root of $J_1(z) = 0$ (including $z = 0$) by $j_{1,k}$, and write $\mu_k = j_{1,k-1}/a$. We use a series of functions

$$h_k(r) = \frac{\sqrt{2}}{a}\frac{J_0(\mu_k r)}{J_0(\mu_k a)} \quad (20)$$

as the bases of the trial function. From the Lommel's integral formulas, the ortho-normalizing condition

$$\int_0^a h_k(r)h_l(r)r\,dr = \delta_{kl} \quad (21)$$

is also proved [9].

B. The Functional

In the following analysis, to treat the cases $m \neq 0$ and $m = 0$ in a unified manner, a new notation

$$F_{m,k}(r) = \begin{cases} g_{m,k}(r), & \text{for } m \neq 0 \\ h_k(r), & \text{for } m = 0 \end{cases} \quad (22)$$

will be used. The ortho-normalizing conditions are then

$$\int_0^a F_{m,k}(r)F_{m,l}(r)r\,dr = \delta_{kl} \quad (23)$$

and the solution $R(r)$ of the variational problem (17) is expressed as

$$R(r) = \sum_{k=1}^N a_k F_{m,k}(r) \quad (24)$$

where a_k should be determined so that the functional in (17) $I[a_1,a_2,\cdots,a_N]$ is minimized.

To determine a_k, we must express each term of (17) in terms of a_k. First, from (24), we obtain

$$R^2(r) = \sum_{k=1}^N\sum_{l=1}^N a_k a_l F_{m,k}(r)F_{m,l}(r). \quad (25)$$

Hence, from the ortho-normalizing conditions (23),

$$\int_0^a R^2(r)r\,dr = \sum_{k=1}^N\sum_{l=1}^N a_k a_l \delta_{kl} \quad (26)$$

in the second term of (17). As for the first term, since $F_{m,k}(a) = \sqrt{2}/a$ holds, we obtain

$$R^2(a) = \frac{2}{a^2}\sum_{k=1}^N\sum_{l=1}^N a_k a_l. \quad (27)$$

The third term of (17) is rewritten, after some computations using the Bessel-function formulas (see the Appendix for the detail), as

$$\int_0^a \left[\left\{\frac{dR(r)}{dr}\right\}^2 + \frac{m^2}{r^2}R^2(r)\right]r\,dr = \sum_{k=1}^N\sum_{l=1}^N a_k a_l X_{kl} \quad (28)$$

where

$$X_{kl} = \begin{cases} \lambda_k^2\delta_{kl} - \dfrac{2m}{a^2}, & \text{for } m \neq 0 \\ \mu_k^2\delta_{kl}, & \text{for } m = 0. \end{cases} \quad (29)$$

Finally, the last term of (17) is given as

$$\int_0^a f(r)R^2(r)r\,dr = \sum_{k=1}^N\sum_{l=1}^N a_k a_l C_{kl} \quad (30)$$

where C_{kl} are parameters given in terms of $f(r)$ as

$$C_{kl} = \begin{cases} \dfrac{2}{a^2 J_m(\lambda_k a)J_m(\lambda_l a)}\displaystyle\int_0^a f(r)J_m(\lambda_k r)J_m(\lambda_l r)r\,dr, & \text{for } m \neq 0 \\ \dfrac{2}{a^2 J_0(\mu_k a)J_0(\mu_l a)}\displaystyle\int_0^a f(r)J_0(\mu_k r)J_0(\mu_l r)r\,dr, & \text{for } m = 0. \end{cases} \quad (31)$$

Putting (26)–(28) and (30) into (17), we may express the functional in terms of a_k as

$$I[a_1,a_2,\cdots,a_N] = \frac{2}{a}\phi_\beta \sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l + (\omega^2\epsilon_1\mu_0 - \beta^2)\sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l \delta_{kl} - \sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l X_{kl} - \omega^2\epsilon_1\mu_0 \sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l C_{kl}. \quad (32)$$

C. Derivation of the Characteristic Equation

To minimize the functional I with respect to all the parameters a_k, the following conditions must be satisfied for all k:

$$\frac{\partial I}{\partial a_k} = \frac{4}{a}\phi_\beta \sum_{l=1}^{N} a_l + 2(\omega^2\epsilon_1\mu_0 - \beta^2)\sum_{l=1}^{N} a_l\delta_{kl} - 2\sum_{l=1}^{N} a_l X_{kl} - 2\omega^2\epsilon_1\mu_0 \sum_{l=1}^{N} a_l C_{kl} = 0. \quad (33)$$

These conditions are reduced to a set of homogeneous linear equations

$$\sum_{l=1}^{N} a_l S_{kl} = 0, \qquad (k = 1,2,\cdots,N) \quad (34)$$

where S_{kl} are given as

$$S_{kl} = \begin{cases} 2(a\phi_\beta + m) + (u^2 - j_{m-1,k}{}^2)\delta_{kl} - \omega^2\epsilon_1\mu_0 a^2 C_{kl}, & (m \neq 0) \\ 2a\phi_\beta + (u^2 - j_{1,k-1}{}^2)\delta_{kl} - \omega^2\epsilon_1\mu_0 a^2 C_{kl}, & (m = 0) \end{cases} \quad (35)$$

$$u = (\omega^2\epsilon_1\mu_0 - \beta^2)^{1/2} a. \quad (36)$$

The relations $\lambda_k a = j_{m-1,k}$ and $\mu_k a = j_{1,k-1}$ have been used in obtaining the previous equations. In order that a nontrivial solution of (34) exists,

$$\det(S_{kl}) = 0 \quad (37)$$

must hold. This equation is the characteristic equation which determines the propagation constant of the optical fiber.

In the previous discussions, the parameter ϕ_β giving the continuity condition at the core–cladding boundary has been left undetermined. In the following we calculate the parameter ϕ_β to make the characteristic equation complete. We assume that the uniform cladding extends to the infinite distance.

1) Case $m \neq 0$: In this case the solution of (12) is given as $R_{\text{clad}}(r) = B_m K_m(wr/a)$, where B_m and K_m denote arbitrary constants and the mth order modified Bessel functions of the second kind, respectively, and

$$w = (\beta^2 - \omega^2\epsilon_2\mu_0)^{1/2} a. \quad (38)$$

Calculating the right-hand side of (13) using formulas of the modified Bessel functions, we obtain two expressions for $a\phi_\beta$:

$$a\phi_\beta = \begin{cases} -m - wK_{m-1}(w)/K_m(w), & \text{for } i = 1, \text{ (case I)} \quad (39a) \\ m - wK_{m+1}(w)/K_m(w), & \text{for } i = 2, \text{ (case II)}. \quad (39b) \end{cases}$$

2) Case $m = 0$: In this case the transverse field in the cladding is expressed as $R_{\text{clad}}(r) = B_0 K_0(wr/a)$, where B_0 is an arbitrary constant. Hence the parameter $a\phi_\beta$ becomes

$$a\phi_\beta = -\frac{wK_1(w)}{K_0(w)}, \qquad \text{(case III)}. \quad (39c)$$

D. Unified Formulation of the Characteristic Equation

To treat the three cases I (39a), II (39b), and III (39c) in a unified manner, we put (39) into (35), and use the notation

$$S_{kl} = \Phi + U_k\delta_{kl} - Y_{kl} \quad (40)$$

where

$$\Phi = \begin{cases} -\dfrac{2wK_n(w)}{K_{n+1}(w)} \cdots \text{case I} & (n \geq 0, i = 1) \quad (41a) \\ 4(n-1) - \dfrac{2wK_n(w)}{K_{n-1}(w)} \cdots \text{case II} & (n \geq 2, i = 2) \quad (41b) \\ -\dfrac{2wK_1(w)}{K_0(w)} \cdots \text{case III} & (n = 1, i = 2) \quad (41c) \end{cases}$$

$$U_k = \begin{cases} (u^2 - j_{n,k}{}^2) \cdots \text{case I} & (42a) \\ (u^2 - j_{n-2,k}{}^2) \cdots \text{case II} & (42b) \\ (u^2 - j_{1,k-1}{}^2) \cdots \text{case III} & (42c) \end{cases}$$

and

$$Y_{kl} = \begin{cases} \dfrac{2\omega^2\epsilon_1\mu_0 a^2}{J_{n+1}(j_{n,k})J_{n+1}(j_{n,l})} \displaystyle\int_0^1 f(ax)J_{n+1}(j_{n,k}x) \cdot J_{n+1}(j_{n,l}x)x\,dx \cdots \text{(I)} & (43a) \\ \dfrac{2\omega^2\epsilon_1\mu_0 a^2}{J_{n-1}(j_{n-2,k})J_{n-1}(j_{n-2,l})} \displaystyle\int_0^1 f(ax)J_{n-1}(j_{n-2,k}x) \cdot J_{n-1}(j_{n-2,l}x)x\,dx \cdots \text{(II)} & (43b) \\ \dfrac{2\omega^2\epsilon_1\mu_0 a^2}{J_0(j_{1,k-1})J_0(j_{1,l-1})} \displaystyle\int_0^1 f(ax)J_0(j_{1,k-1}x) \cdot J_0(j_{1,l-1}x)x\,dx \cdots \text{(III)}. & (43c) \end{cases}$$

From (37) and (40), we obtain

$$\begin{vmatrix} \Phi+U_1-Y_{11} & \Phi-Y_{12} & \cdots & \Phi-Y_{1N} \\ \Phi-Y_{21} & \Phi+U_2-Y_{22} & \cdots & \Phi-Y_{2N} \\ \vdots & \vdots & & \vdots \\ \Phi-Y_{N1} & \Phi-Y_{N2} & \cdots & \Phi+U_N-Y_{NN} \end{vmatrix} = 0. \tag{44}$$

Equation (44) may be rewritten to

$$D + \Phi \sum_{k=1}^{N} \sum_{l=1}^{N} Z_{kl} = 0 \tag{45}$$

where D is given as

$$D = \begin{vmatrix} U_1 - Y_{11} & -Y_{12} & \cdots & -Y_{1N} \\ -Y_{21} & U_2 - Y_{22} & \cdots & -Y_{2N} \\ \vdots & \vdots & & \vdots \\ -Y_{N1} & -Y_{N2} & \cdots & U_N - Y_{NN} \end{vmatrix} \tag{46}$$

and Z_{kl} is the cofactor of the element $d_{kl}(= U_k\delta_{kl} - Y_{kl})$ of matrix D. From (45), the characteristic equation (the relation between u and w) of the optical fiber having an arbitrary permittivity profile may be written in a unified form

$$-\frac{1}{\Phi} = \frac{1}{D} \sum_{k=1}^{N} \sum_{l=1}^{N} Z_{kl}. \tag{47}$$

IV. ANALYSIS OF THE PROPAGATION CHARACTERISTICS

A. Homogeneous-Core Fibers

We first consider this simplest case with results that can be compared with analytic solutions reported previously [10]–[12]. In this case $f(r) = 0$ in (1); consequently, from (43) $Y_{kl} = 0$ for all k and l. Hence we obtain from (46)

$$D = \prod_{k=1}^{N} U_k \tag{48}$$

and Z_{kl} are given as

$$Z_{kl} = \begin{cases} \prod_{k=1;k\neq l}^{N} U_k, & \text{for } k = l \\ 0, & \text{for } k \neq l. \end{cases} \tag{49}$$

Putting (48) and (49) into (47) and setting $N \to \infty$, we obtain the characteristic equation for the homogeneous-core fiber as

$$-\frac{1}{\Phi} = \sum_{k=1}^{\infty} \frac{1}{U_k}. \tag{50}$$

We first consider the above case I. From one of the Bessel function formulas [9], we obtain

$$\sum_{k=1}^{\infty} \frac{1}{U_k} = \sum_{k=1}^{\infty} \frac{1}{(u^2 - j_{n,k}^2)} = -\frac{J_{n+1}(u)}{2uJ_n(u)}. \tag{51}$$

Hence, from (41a), (50), and (51), the characteristic equation is given as

$$\frac{K_{n+1}(w)}{wK_n(w)} = -\frac{J_{n+1}(u)}{uJ_n(u)}. \tag{52}$$

Referring to the classification of modes given by Snyder [12], it is seen that (52) is identical to the characteristic equation of the TE_{0l} mode if $n = 0$, and that of the EH_{nl} mode if $n \geq 1$.

Next, for the above case II, using (41b), (42b), and (50), we obtain

$$\frac{K_{n-1}(w)}{wK_n(w)} = \frac{J_{n-1}(u)}{uJ_n(u)} \tag{53}$$

which is the characteristic equation of the HE_{nl} mode ($n \geq 2$) given by Snyder [12]. For the case III, from (41c), (42c), and (50), the characteristic equation is given as

$$\frac{K_0(w)}{wK_1(w)} = \frac{J_0(u)}{uJ_1(u)} \tag{54}$$

which is identical to the HE_{1l}-mode equation [12].

Thus the validity of the present analysis has partly been proved by the comparison with analytic solutions for the special case. We should here note that when we solve this sort of problem by using the Rayleigh–Ritz method, we obtain a set of characteristic equations directly but cannot tell to what kinds of mode they correspond. In the present case, the nature of the modes could be known by comparing the equations with the analytic solutions. In more general cases as described in the following, the nature of a mode can be known only after the corresponding field distribution has been computed.

B. Inhomogeneous-Core Fibers

To treat the problem more generally, we go back to (1), or an equivalent expression in terms of the refractive index:

$$n^2(r) = n_1^2[1 - f(r)], \qquad 0 \leq r \leq a \tag{55}$$

where n_1 denotes the refractive index upon the axis. We introduce here a parameter Δ which represents the relative difference of the refractive indexes in the core and cladding:

$$\Delta = \frac{(n_1^2 - n_2^2)}{2n_1^2} \tag{56}$$

where n_2 denotes the refractive index of the cladding. Note that when $\Delta \ll 1$, we may approximate $\Delta \doteqdot (n_1 - n_2)/n_1$.

In principle, the method of analysis described above may be applied to any refractive-index distribution. However, actually an entirely arbitrary, or extremely peculiar distributions are not used. In the following discussions, to be practical, we deal with "a set of various refractive-index profiles including the homogeneous-core and the quadratic refractive-index distribution as two special cases."

The quadratic distribution may be expressed by

$$f(r) = 2\Delta r^2/a^2. \tag{57}$$

Hence, we may use the following formula to express refractive-index distributions "between" the homogeneous and the quadratic cases:

$$f(r) = 2\Delta \frac{\sinh^2 (Pr/a)}{\sinh^2 (P)}. \tag{58}$$

When $P \rightarrow 0$ this index profile approaches the quadratic distribution, whereas when $P \rightarrow \infty$ to the homogeneous-core case.

To express refractive-index distributions "sharper than" the quadratic distribution, we use

$$f(r) = 2\Delta \frac{\tanh^2 (Qr/a)}{\tanh^2 (Q)} \tag{59}$$

which, when $Q \rightarrow 0$ approaches the quadratic distribution, and when $Q \rightarrow \infty$ represents the "delta-function-like" refractive-index distribution. Those various refractive-index profiles are illustrated in Fig. 2.

Next, following reference [11], we introduce a parameter v which is defined as

$$v = (u^2 + w^2)^{1/2}. \tag{60}$$

From (36), (38), (56), and (60), we obtain

$$\omega^2 \epsilon_1 \mu_0 a^2 = \frac{v^2}{2\Delta} \tag{61}$$

which shows that v is proportional to the frequency. Accordingly, v is often called the normalized frequency. Using the parameter v we may rewrite (43a)–(43c) as

$$Y_{kl} = \begin{cases} 2v^2 \int_0^1 \rho(x) \dfrac{J_{n+1}(j_{n,k}x) J_{n+1}(j_{n,l}x)}{J_{n+1}(j_{n,k}) J_{n+1}(j_{n,l})}\, x\, dx \cdots \text{case I} & \text{(62a)} \\ 2v^2 \int_0^1 \rho(x) \dfrac{J_{n-1}(j_{n-2,k}x) J_{n-1}(j_{n-2,l}x)}{J_{n-1}(j_{n-2,k}) J_{n-1}(j_{n-2,l})}\, x\, dx \cdots \text{case II} & \text{(62b)} \\ 2v^2 \int_0^1 \rho(x) \dfrac{J_0(j_{1,k-1}x) J_0(j_{1,l-1}x)}{J_0(j_{1,k-1}) J_0(j_{1,l-1})}\, x\, dx \cdots \text{case III} & \text{(62c)} \end{cases}$$

where $x = r/a$ and $\rho(x) = (1/2\Delta) f(ax)$.

The previous equations tell that Y_{kl} are proportional to the square of the normalized frequency v. Hence (47) gives implicitly the relation between u and v because $\Phi = \text{func}\,(w)$ and $w^2 = v^2 - u^2$. Since the parameters a and Δ characterizing the properties of the fiber do not appear in the equation, the propagation characteristics of the fiber may be analyzed generally upon the basis of (47).

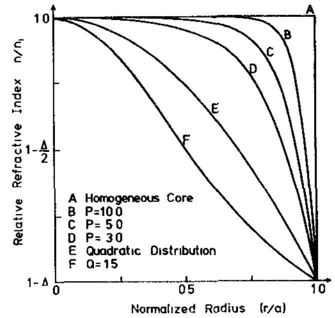

Fig. 2. Various refractive-index distributions used in the numerical analysis. B, C, and D: $n/n_1 = 1 - \Delta[\sinh^2 (Pr/a)/\sinh^2 (P)]$. E: $n/n_1 = 1 - \Delta(r/a)^2$. F: $n/n_1 = 1 - \Delta[\tanh^2 (Qr/a)/\tanh^2 (Q)]$.

C. Fields in the Fiber

When the relative refractive-index difference Δ and the core radius a are given, we may determine the propagation constant β_l from the characteristic equation (47), and further determine the values of u, v, and S_{kl} $(k,l = 1,2, \cdots, N)$. From those S_{kl}, the coefficients a_k in (24) are given as the solution of the following simultaneous equations of the order $(N-1)$:

$$\begin{pmatrix} S_{1,1} & \cdots & S_{1,l-1} & S_{1,l+1} & \cdots & S_{1,N} \\ \vdots & & \vdots & \vdots & & \vdots \\ S_{l-1,1} & \cdots & S_{l-1,l-1} & S_{l-1,l+1} & \cdots & S_{l-1,N} \\ S_{l+1,1} & \cdots & S_{l+1,l-1} & S_{l+1,l+1} & \cdots & S_{l+1,N} \\ \vdots & & \vdots & \vdots & & \vdots \\ S_{N,1} & \cdots & S_{N,l-1} & S_{N,l+1} & \cdots & S_{N,N} \end{pmatrix} \begin{pmatrix} A_{1,l} \\ \vdots \\ A_{l-1,l} \\ A_{l+1,l} \\ \vdots \\ A_{N,l} \end{pmatrix} = - \begin{pmatrix} S_{1,l} \\ \vdots \\ S_{l-1,l} \\ S_{l+1,l} \\ \vdots \\ S_{N,l} \end{pmatrix} \tag{63}$$

where $A_{kl} = a_k/a_l$ $(k = 1,2,\cdots,l-1, l+1,\cdots,N)$. By putting these a_k thus obtained into (24), the transverse fields $\Psi^{(i)}$ $(i = 1,2)$ of the lth mode is determined except for a constant factor. If we use A_{kl} instead of a_k, we may express the transverse field in the core as

$$\Psi^{(1)}(r) = R_{l0} \sum_{k=1}^{N} A_{kl} \frac{J_{n+1}(j_{n,k} r/a)}{J_{n+1}(j_{n,k})} \cdots \text{case I} \tag{64a}$$

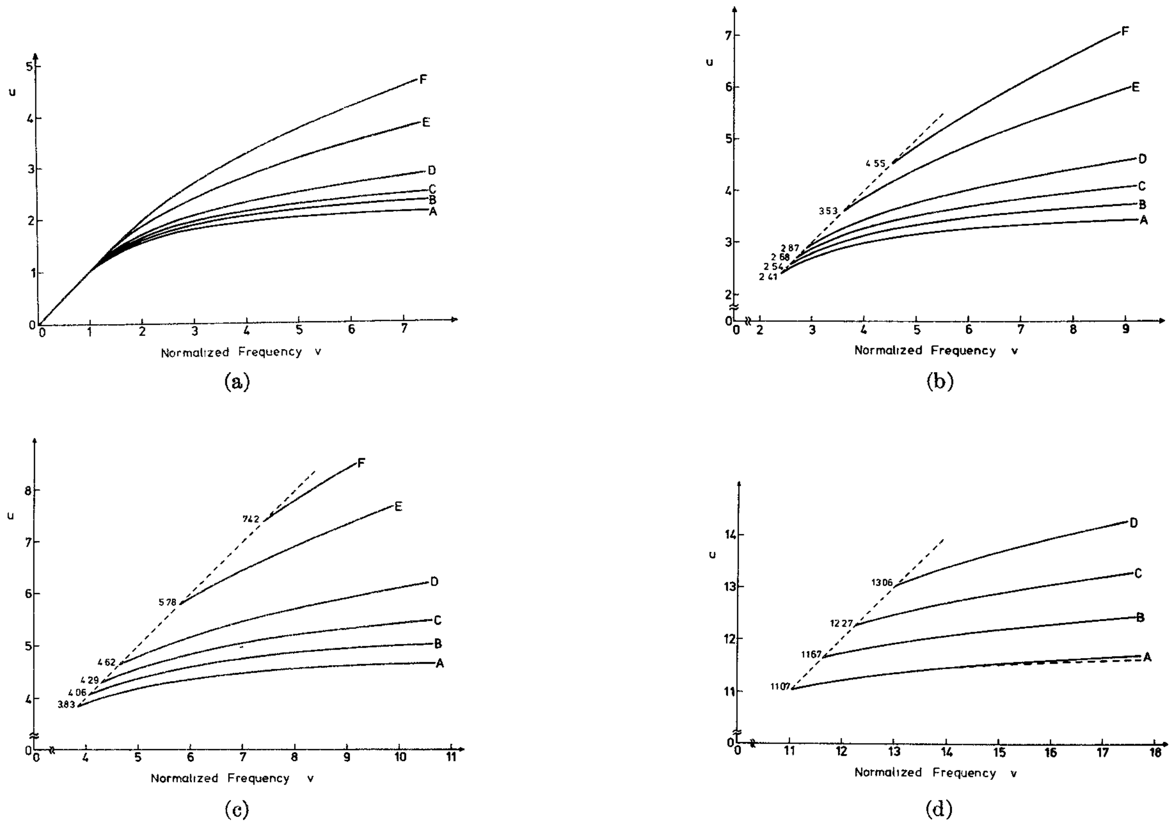

Fig. 3. Propagation characteristics for various refractive-index distributions. (a) HE_{11} mode. (b) TE_{01} mode. (c) HE_{31} mode. (d) EH_{42} mode. [The dotted curve in (d) is the characteristic curve obtained by the analytical solution ($n = 4$, $l = 2$ in (52)).]

$$\Psi^{(2)}(r) = \begin{cases} R_{l0} \sum_{k=1}^{N} A_{kl} \dfrac{J_{n-1}(j_{n-2,k} r/a)}{J_{n-1}(j_{n-2,k})} \cdots \text{case II} & \text{(64b)} \\[2ex] R_{l0} \sum_{k=1}^{N} A_{kl} \dfrac{J_0(j_{1,k-1} r/a)}{J_0(j_{1,k-1})} \cdots \text{case III} & \text{(64c)} \end{cases}$$

where $A_{ll} = 1$ and $R_{l0} = \sqrt{2}\, a_l/a$, and the field in the cladding as

$$\Psi_{\mathrm{clad}}^{(1)}(r) = \Psi^{(1)}(a) \frac{K_{n+1}(wr/a)}{K_{n+1}(w)} \cdots \text{case I} \qquad \text{(65a)}$$

$$\Psi_{\mathrm{clad}}^{(2)}(r) = \begin{cases} \Psi^{(2)}(a) \dfrac{K_{n-1}(wr/a)}{K_{n-1}(w)} \cdots \text{case II} & \text{(65b)} \\[2ex] \Psi^{(2)}(a) \dfrac{K_0(wr/a)}{K_0(w)} \cdots \text{case III.} & \text{(65c)} \end{cases}$$

V. RESULTS OF COMPUTER ANALYSIS AND DISCUSSIONS

The computed propagation characteristics and the transverse field distributions for the various refractive-index profiles illustrated in Fig. 2 are shown in Figs. 3 and 4. Fig. 3(a)–(d) shows the normalized eigenvalue u as a function of the normalized frequency v for various refractive-index profiles. Fig. 4(a)–(d) shows the field distributions for various refractive-index profiles, the constant R_{l0} in (64) being determined so that the transmitted power is constant.

The characteristic curves and field distributions obtained with the present variational-method analysis employing expansion terms up to $N = 10$ show good agreement with the analytic solutions reported previously in the literature [10]–[12]. In most cases the difference in u is too small to show in Figs. 3 and 4 (below 3 percent). An exception is the lowest curve in Fig. 3(d), where a relatively big difference from the analytic solution (dotted curve) is found. This suggests that in the abrupt boundary case more expansion terms should be computed for larger v.

Since no cutoff condition is present for the HE_{11} mode, it is the lowest transmission mode of the optical fiber. The numerals in Fig. 3(b)–(d) indicate the cutoff values of v for each mode. The cutoff v value of the TE_{01} mode gives the single-mode limit of the fiber because the TE_{01} mode is the second lowest mode. For an optical fiber having a quadratic refractive-index distribution, this limit is given by $v = 3.53$. This value agrees with the value obtained previously by Dil's analysis [6].

VI. CONCLUSION

A variational method analysis has been proposed to determine the propagation characteristics of an optical fiber consisting of a core with an arbitrary refractive-index distribution and a uniform cladding. Some results of the

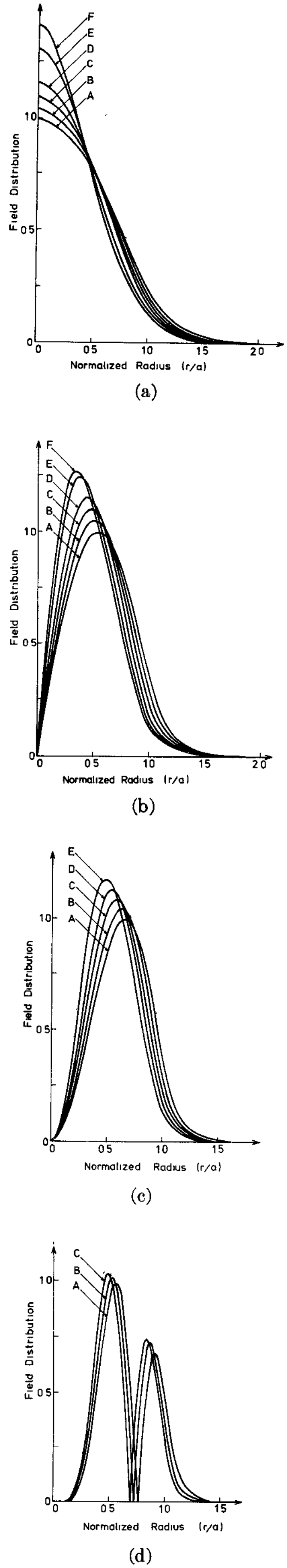

Fig. 4. The transverse field patterns for various refractive-index distributions. (a) HE_{11} mode. (b) TE_{01} mode. (c) HE_{31} mode. (d) EH_{42} mode.

analysis are shown which support the validity of the analysis.

APPENDIX

DERIVATION OF (29)

From (24), X_{kl} in (28) is expressed as

$$X_{kl} = \int_0^a \left[\left(\frac{dF_{m,k}}{dr}\right)\left(\frac{dF_{m,l}}{dr}\right) + \left(\frac{m}{r} F_{m,k}\right)\left(\frac{m}{r} F_{m,l}\right)\right] r\, dr. \tag{A.1}$$

In the case $m \neq 0$, from (17) and the Bessel-function formulas,

$$\frac{dF_{m,k}}{dr} = \frac{\lambda_k}{\sqrt{2}\, a J_m(\lambda_k a)} \{J_{m-1}(\lambda_k r) - J_{m+1}(\lambda_k r)\}$$

$$\frac{m}{r} F_{m,k} = \frac{\lambda_k}{\sqrt{2}\, a J_m(\lambda_k a)} \{J_{m-1}(\lambda_k r) + J_{m+1}(\lambda_k r)\}. \tag{A.2}$$

Using Lommel's integral formulas, X_{kl} is given as

$$X_{kl} = \begin{cases} \lambda_k^2 - \dfrac{2m}{a^2}, & (k = l) \\ -\dfrac{2m}{a^2}, & (k \neq l). \end{cases} \tag{A.3}$$

Next, in the case $m = 0$, using (20) we readily obtain

$$X_{kl} = \mu_k^2 \delta_{kl}. \tag{A.4}$$

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